

Weak metacirculants of odd prime power order

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Abstract

Metacirculants are a basic and well-studied family of vertex-transitive graphs, and weak metacirculants are generalizations of them. A graph is called a weak metacirculant if it has a vertex-transitive metacyclic automorphism group. This paper is devoted to the study of weak metacirculants with odd prime power order. We first prove that a weak metacirculant of odd prime power order is a metacirculant if and only if it has a vertex-transitive split metacyclic automorphism group. We then prove that for any odd prime p and integer $\ell \geq 4$, there exist weak metacirculants of order p^ℓ which are Cayley graphs but not Cayley graphs of any metacyclic group; this answers a question in [C. H. Li, S. J. Song and D. J. Wang, A characterization of metacirculants, J. Combin. Theory Ser. A 120 (2013) 39–48]. We construct such graphs explicitly by introducing a construction which is a generalization of generalized Petersen graphs. Finally, we determine all smallest possible metacirculants of odd prime power order which are Cayley graphs but not Cayley graphs of any metacyclic group.

Keywords: metacirculant, weak metacirculant, Cayley graph, metacyclic group

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1 Introduction

Let $m \geq 1$ and $n \geq 2$ be integers. A graph $\Gamma = (V(\Gamma), E(\Gamma))$ of order mn is called [13] an (m, n) -metacirculant graph (in short (m, n) -metacirculant) if it has an automorphism σ of order n such that $\langle \sigma \rangle$ is semiregular on $V(\Gamma)$, and an automorphism τ normalizing $\langle \sigma \rangle$ and cyclically permuting the m orbits of $\langle \sigma \rangle$ such that τ has a cycle of size m in its cycle decomposition. A graph is called a *metacirculant* if it is an (m, n) -metacirculant for some m and n . It follows from this definition that a metacirculant Γ has an automorphism group $\langle \sigma, \tau \rangle$ which is metacyclic and transitive on $V(\Gamma)$. In general, a group G is called *metacyclic* if it contains a cyclic normal subgroup N such that G/N is cyclic. In other words, a metacyclic group G is an extension of a cyclic group $N \cong C_n$ by a cyclic group $G/N \cong C_m$, written $G \cong C_n.C_m$. If this extension is split, namely $G \cong C_n : C_m$, then G is called a *split metacyclic group*.

Introduced by Alspach and Parsons [1], metacirculants form a basic class of vertex-transitive graphs. As a generalization of metacirculants, Marušič and Šparl [13] introduced the following concept: A graph is called a *weak metacirculant* if it has a vertex-transitive metacyclic automorphism group. In [12], Li et al. divided the class of weak metacirculants into the following two

subclasses: A weak metacirculant is called a *split weak metacirculant* or *non-split weak metacirculant* according to whether or not its full automorphism group contains a vertex-transitive split metacyclic subgroup. In [12], Li et al. studied the relationship between metacirculants and weak metacirculants. Among other results they proved that every metacirculant is a split weak metacirculant (see [12, Lemma 2.2]), but it was unknown whether the converse of this statement is true. So the following question arises naturally.

Question A Is it true that any split weak metacirculant is a metacirculant?

In this paper we first give a positive answer to this question for split weak metacirculants of odd prime power order, as stated in the following result.

Theorem 1.1 *A connected weak metacirculant with order an odd prime power is a metacirculant if and only if it is a split weak metacirculant.*

Question A is open for split weak metacirculants of order not an odd prime power; in fact, there is no result concerning Question A in the literature in this case as far as we know.

Obviously, any Cayley graph of a metacyclic group is a weak metacirculant; such a graph is called a *weak metacirculant Cayley graph* (see [12, p.41]). Weak metacirculant Cayley graphs form a large class of weak metacirculants. However, not every weak metacirculant is a Cayley graph. For example, the Petersen graph is a $(2, 5)$ -metacirculant but not a Cayley graph. The following question was posed by Pan [14, p.15] and Li et al. [12, p.41] independently.

Question B Is it true that a weak metacirculant which is a Cayley graph of some (not necessarily metacyclic) group must be a weak metacirculant Cayley graph?

Our second main result gives a negative answer to this question.

Theorem 1.2 *Let p be an odd prime. Then for any integer $\ell \geq 4$ there exists a weak metacirculant of order p^ℓ which is a Cayley graph but not a weak metacirculant Cayley graph.*

Moreover, the smallest possible order and valency of a weak metacirculant with order a power of p which is a Cayley graph but not a weak metacirculant Cayley graph are p^4 and $2p+2$, respectively.

The third main result in this paper is the following classification of connected metacirculants of order p^4 and valency $2p+2$, where p is an odd prime. The graph $\text{MP}_{p^3, p^2, p^2, \lambda}$ involved in the classification will be defined in Definition 5.1; it belongs to a large family of graphs that contains all generalized Petersen graphs as a proper subfamily.

Theorem 1.3 *Let p be an odd prime. Let Γ be a connected metacirculant of order p^4 and valency $2p+2$. Then one of the following holds:*

- (a) Γ is a metacirculant Cayley graph;
- (b) Γ is not a Cayley graph;
- (c) Γ is isomorphic to $\text{MP}_{p^3, p^2, p^2, \lambda}$ for some element λ of $\mathbb{Z}_{p^3}^*$ with order p^2 .

This result seems to suggest that most weak metacirculants which are Cayley graphs are weak metacirculant Cayley graphs. Nevertheless, more research is needed to find out whether this is indeed the case.

The rest of this paper is organized as follows. In the next section we will collect some basic definitions on permutation groups, Cayley graphs and vertex-transitive graphs. In section 3,

we will give the proof of Theorem 1.1 after presenting a few results on p -groups. In section 4, we will prove that any weak metacirculant of order an odd prime power p^n must be a weak metacirculant Cayley graph if its valency is less than $2p+2$ or its order is at most p^3 . This result will be used in the proof of Theorem 1.2, which will be given in section 6. Another preparation for the proof of Theorem 1.2 is the construction of multilayer generalized Petersen graphs, which will be introduced in section 5. The proof of Theorem 1.3 will be given in section 7.

2 Preliminaries

2.1 Definitions and notation

Given a group G , denote by 1_G , $\text{Aut}(G)$, $Z(G)$, $\Phi(G)$ and G' the identity element, full automorphism group, center, Frattini subgroup and derived subgroup of G , respectively. Denote by $o(x)$ the order of an element x of G . For a subgroup H of G , denote by $C_G(H)$, $N_G(H)$ the centralizer and normalizer of H in G , respectively. Of course $C_G(H)$ is normal in $N_G(H)$, and the well-known N/C theorem asserts that the quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\text{Aut}(H)$. Given a p -group G of exponent p^e , where p is a prime and $e \geq 1$ an integer, for each integer s between 0 and e , set

$$\Omega_s(G) = \langle g \in G \mid g^{p^s} = 1_G \rangle.$$

A *block of imprimitivity* of a permutation group G on a set Ω is a subset Δ of Ω with $1 < |\Delta| < |\Omega|$ such that for any $g \in G$, either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$. In this case the *blocks* Δ^g , $g \in G$ form a G -invariant partition of Ω .

We reserve C_n for the cyclic group of order n , \mathbb{Z}_n for the ring of integers modulo n , and \mathbb{Z}_n^* for the multiplicative group of units of \mathbb{Z}_n consisting of integers coprime to n .

All graphs in this paper are finite, simple and undirected. Given a graph Γ and $u, v \in V(\Gamma)$, denote by $u \sim v$ the relation that u is adjacent to v in Γ , by $\{u, v\}$ the edge between u and v , and by (u, v) the arc from u to v . Denote by $\Gamma(v)$ the neighbourhood of v , and by $\Gamma[B]$ the subgraph of Γ induced by a subset B of $V(\Gamma)$. An s -cycle in Γ , denoted by Cyc_s , is an $(s+1)$ -tuple of pairwise distinct vertices (v_0, v_1, \dots, v_s) such that $\{v_{i-1}, v_i\} \in E(\Gamma)$ for $1 \leq i \leq s$ and $\{v_s, v_0\} \in E(\Gamma)$. Denote by K_n the complete graph of order n , and $K_{n,n}$ the complete bipartite graph with biparts of cardinality n . The *lexicographic product* of a graph Γ_1 by a graph Γ_2 , denoted by $\Gamma_1[\Gamma_2]$, is the graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ such that $\{(x_1, x_2), (y_1, y_2)\} \in E(\Gamma_1[\Gamma_2])$ if and only if either $\{x_1, y_1\} \in E(\Gamma_1)$, or $x_1 = y_1$ and $\{x_2, y_2\} \in E(\Gamma_2)$.

The full automorphism group of a graph Γ is denoted by $\text{Aut}(\Gamma)$. Γ is called G -vertex-transitive (respectively, G -edge-transitive) if $G \leq \text{Aut}(\Gamma)$ and G is transitive on $V(\Gamma)$ (respectively, $E(\Gamma)$); in this case G is said to be a vertex-transitive (respectively, edge-transitive) subgroup of Γ . Γ is *vertex-transitive* (respectively, *edge-transitive*) if it is $\text{Aut}(\Gamma)$ -vertex-transitive (respectively, $\text{Aut}(\Gamma)$ -edge-transitive). G -arc-transitive graphs and *arc-transitive* graphs are understood similarly. Given a G -vertex-transitive graph Γ and a G -invariant partition \mathcal{B} of $V(\Gamma)$, the *quotient graph* of Γ with respect to \mathcal{B} , denoted by $\Gamma_{\mathcal{B}}$, is defined as the graph with vertex set \mathcal{B} such that, for distinct $B, C \in \mathcal{B}$, B is adjacent to C if and only if there exist $u \in B$ and $v \in C$ which are adjacent in Γ . In particular, for a normal subgroup N of G , the set \mathcal{B} of orbits of N on $V(\Gamma)$ is a G -invariant partition of $V(\Gamma)$, and in this case we use Γ_N in place of $\Gamma_{\mathcal{B}}$.

2.2 Cayley graphs

Given a finite group G and an inverse-closed subset $S \subseteq G \setminus \{1_G\}$, the *Cayley graph* $\text{Cay}(G, S)$ of G with respect to S is the graph with vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. It

is well known that the right regular representation $R(G) = \{R(g) \mid g \in G\}$ of G is a subgroup of $\text{Aut}(\text{Cay}(G, S))$, where $R(g)$ is the permutation of G defined by $R(g) : x \mapsto xg$ for $x \in G$. In [9], Godsil proved that the normalizer of $R(G)$ in $\text{Aut}(\text{Cay}(G, S))$ is $R(G) \rtimes \text{Aut}(G, S)$, where $\text{Aut}(G, S)$ is the group of automorphisms of G fixing S setwise. In the case when $R(G)$ is normal in $\text{Aut}(\text{Cay}(G, S))$, $\text{Cay}(G, S)$ is called [20] a *normal Cayley graph*. The reader is referred to [8] for recent results on normal Cayley graphs.

It is well known that a graph Γ is isomorphic to a Cayley graph if and only if it has an automorphism group acting regularly on its vertex set (see [4, Lemma 16.3]). In general, a permutation group G on a set Ω is called *semiregular* on Ω if $G_\alpha = 1_G$ for every $\alpha \in \Omega$, and *regular* on Ω if G is transitive and semiregular on Ω , where G_α is the *stabilizer* of α in G , defined as the subgroup of G consisting of those elements of G which fix α .

2.3 Coset graphs

Let G be a finite group, H a subgroup of G , and D the union of some double-cosets HgH with $g \notin H$ such that $D = D^{-1}$. The *coset graph* $\Gamma = \text{Cos}(G, H, D)$ of G with respect to H and D is defined as the graph with vertex set $V(\Gamma) = [G : H]$, the set of right cosets of H in G , and edge set $E(\Gamma) = \{\{Hg, Hd\} \mid g \in G, d \in D\}$. It is easy to see that Γ is well defined and has valency $|D|/|H|$. Further, Γ is connected if and only if D generates G . In the special case when $H = 1_G$, Γ is the Cayley graph of G with respect to D . Denote by R_H the right multiplication action of G on $V(\Gamma) = [G : H]$, defined by $R_H(g) : Hx \mapsto Hxg$, $Hx \in [G : H]$. (In particular, $R_{1_G}(G)$ is the right regular representation $R(G)$ of G .) Then R_H is transitive on $V(\Gamma)$, and R_H is faithful on $V(\Gamma)$ if and only if H is *core-free* in G , that is, $\bigcap_{g \in G} H^g = 1_G$. It is easy to see that $R_H(G) \leq \text{Aut}(\Gamma)$. Hence Γ is vertex-transitive. In [17], Sabidussi proved that all vertex-transitive graphs can be constructed this way up to isomorphism.

Proposition 2.1 *The coset graph $\text{Cos}(G, H, D)$ constructed above is G -vertex-transitive. Conversely, if Γ is a G -vertex-transitive graph, then it is isomorphic to a coset graph $\text{Cos}(G, H, D)$, where $H = G_\alpha$ for a fixed $\alpha \in V(\Gamma)$ and D consists of all elements of G which map α to one of its neighbours.*

The following results are well known; see for example [11, Lemma 2.1].

Lemma 2.2 *Let G be a group and H a core-free subgroup of G . Take $g \in G \setminus H$ and let $\Gamma = \text{Cos}(G, H, H\{g, g^{-1}\}H)$. Then the following hold:*

- (a) Γ is G -edge-transitive;
- (b) Γ is G -arc-transitive if and only if $HgH = Hg^{-1}H$;
- (c) Γ is connected if and only if $G = \langle H, g \rangle$;
- (d) the valency of Γ is equal to $|H : H \cap H^g|$ if $HgH = Hg^{-1}H$, or $2|H : H \cap H^g|$ if $HgH \neq Hg^{-1}H$.

3 Proof of Theorem 1.1

3.1 Some results on p -groups

In order to prove Theorem 1.1, we first present a few results on p -groups. The following result is due to Newman, Xu and Zhang; see [21, Theorem 2.1].

Lemma 3.1 *Let p be an odd prime and G a metacyclic p -group. Then G has representation*

$$G = \langle a, b \mid a^{p^{r+s+u}} = 1_G, b^{p^{r+s+t}} = a^{p^{r+s}}, b^{-1}ab = a^{1+p^r} \rangle$$

for some nonnegative integers r, s, t, u such that $r \geq 1, r \geq u$. Moreover, different values of the parameters r, s, t, u satisfying these conditions give rise to non-isomorphic metacyclic p -groups, and G is non-split if and only if $stu \neq 0$.

The following result can be easily proved (see, for example, [2, Exercise 85]).

Lemma 3.2 *Let G be a noncyclic metacyclic p -group. If $p > 2$, then $\Omega_1(G) \cong C_p \times C_p$.*

A p -group G is said to be p^r -abelian if $(xy)^{p^r} = x^{p^r}y^{p^r}$ for any $x, y \in G$.

Lemma 3.3 *Any metacyclic p -group G with $p > 2$ is p^ℓ -abelian, where $p^\ell = |G'|$.*

Proof. By [2, Theorem 7.1 (c)], G is regular, and then by [10, III, 10.8(g)], G is p^ℓ -abelian. \square

Lemma 3.4 *Let p be an odd prime. Let $G = \langle \sigma \rangle : \langle \tau \rangle \cong C_{p^m} : C_{p^n}$ with $m \geq n \geq 1$. For any $1_G \neq g \in G$, if $\langle g \rangle \cap \langle \sigma \rangle = 1_G$, then there exists $\tau' \in G$ such that $\langle \tau' \rangle \cong C_{p^n}$ and $g \in \langle \tau' \rangle$.*

Proof. We make induction on the order $|G|$ of G . Clearly, $|G| = p^{m+n} \geq p^2$. Suppose that $|G| = p^2$. Then $m = n = 1$ and $G = \langle \sigma \rangle : \langle \tau \rangle \cong C_p \times C_p$. For any $1_G \neq g \in G$, if $\langle g \rangle \cap \langle \sigma \rangle = 1_G$, then $\langle g \rangle \cong C_p$ and $G = \langle \sigma \rangle \times \langle g \rangle$, as required.

In what follows we assume that $|G| > p^2$. Let $1_G \neq g \in G$ be such that $\langle g \rangle \cap \langle \sigma \rangle = 1_G$. Since $G/\langle \sigma \rangle \cong C_{p^n}$, g has order at most p^n . If the order $o(g)$ of g is equal to p^n , then the result is clearly true. Assume $o(g) = p^k < p^n$. Then $n > k \geq 1$.

Suppose first that $m > n$. Then $g, \tau \in \Omega_{m-1}(G)$. Since $G/\langle \sigma \rangle \cong C_{p^n}$, we have $G' \leq \langle \sigma \rangle$. Clearly, $G' \neq \langle \sigma \rangle$, so $G' \leq \langle \sigma^p \rangle \cong \mathbb{Z}_{p^{m-1}}$. By Lemma 3.3, G is p^{m-1} -abelian. This implies that $\Omega_{m-1}(G)$ contains no elements of order greater than p^{m-1} , and consequently, $\Omega_{m-1}(G) < G$. Clearly, $\sigma^p \in \Omega_{m-1}(G)$, so $\Omega_{m-1}(G) = \langle \sigma^p \rangle : \langle \tau \rangle \cong C_{p^{m-1}} : C_{p^n}$. Since $m-1 \geq n$, by induction there exists $\tau' \in \Omega_{m-1}(G)$ such that $\langle \tau' \rangle \cong C_{p^n}$ and $g \in \langle \tau' \rangle$, as required.

Now suppose that $m = n$. Since $p > 2$, $\text{Aut}(C_{p^m}) \cong C_{p^{m-1}(p-1)}$. Then τ induces an automorphism by conjugation of $\langle \sigma \rangle$ of order at most p^{m-1} . This implies that $\tau^{p^{m-1}}$ commutes with σ , and so $\tau^{p^{m-1}}$ is in the center of G . Since $p > 2$, by Lemma 3.2 we have $\Omega_1(G) \cong C_p \times C_p$ and so $\Omega_1(G) = \langle \sigma^{p^{m-1}} \rangle \times \langle \tau^{p^{m-1}} \rangle \cong C_p \times C_p$.

Let $N = \langle \tau^{p^{m-1}} \rangle$. Then $G/N \cong C_{p^m} : C_{p^{m-1}}$. If $gN = N$, then $g \in N$ and $g \in \langle \tau \rangle$, and so the result holds.

Assume that $gN \neq N$ in the sequel. If $\langle gN \rangle \cap \langle \sigma N \rangle = N$, then by induction, $gN \in \langle \tau'N \rangle$ for some $\tau' \in G$ such that $\langle \tau'N \rangle \cap \langle \sigma N \rangle = N$ and $\langle \tau'N \rangle \cong C_{p^{m-1}}$. So $g \in \langle \tau', N \rangle$. If $\langle \tau' \rangle \cap N = 1_G$, then $\Omega_1(G) \leq \langle \tau', N \rangle$ and so $\langle \sigma^{p^{m-1}}N \rangle = \Omega_1(G)/N \leq \langle \tau', N \rangle/N = \langle \tau'N \rangle$, a contradiction. Thus $\langle \tau' \rangle \cap N \neq 1_G$, and hence $N \leq \langle \tau' \rangle$. This implies that $g \in \langle \tau' \rangle \cong C_{p^n}$, as required.

Suppose that $\langle gN \rangle \cap \langle \sigma N \rangle \neq N$. Then $g^iN = \sigma^jN \neq N$ for some i, j . It then follows that $g^i = \sigma^j(\tau^{p^{m-1}})^k$ for some k . If $p \mid k$, then $g^i = \sigma^j \neq 1_G$, a contradiction. Thus, $(p, k) = 1$. Let $j = p^\ell j'$ be such that $(j', p) = 1$. Then $\ell \leq m-1$. If $\ell < m-1$, then σ^j has order at least p^2 and so $g^{p^\ell} = \sigma^{pj} = \sigma^{p^{\ell+1}j'} \neq 1_G$, which contradicts the fact that $\langle g \rangle \cap \langle \sigma \rangle = 1_G$. Hence, $\ell = m-1$, and so $g^i = (\sigma^{j'}\tau^k)^{p^{m-1}}$ as G is p^{m-1} -abelian.

Since $(k, p) = 1$, we have $G = \langle \sigma, \sigma^{j'}\tau^k \rangle$, and since $1 \neq g^i \in \langle \sigma^{j'}\tau^k \rangle$ and $\langle \sigma \rangle \cap \langle g \rangle = 1_G$, we have $\langle \sigma \rangle \cap \langle \sigma^{j'}\tau^k \rangle = 1_G$. This implies that $G = \langle \sigma \rangle : \langle \sigma^{j'}\tau^k \rangle \cong C_{p^m} : C_{p^m}$ and so $\langle \sigma^{j'}\tau^k \rangle \cong C_{p^m}$.

If g has order p , then $\langle g \rangle = \langle g^i \rangle \leq \langle \sigma^{j'} \tau^k \rangle$, as required. If g has order p^t with $t > 1$, then let $M = \langle g^{p^{t-1}} \rangle = \langle (\sigma^{j'} \tau^k)^{p^{m-1}} \rangle$. Clearly, $G/M \cong C_{p^m} : C_{p^{m-1}}$ and $\langle gM \rangle \cap \langle \sigma M \rangle = M$. By induction, $gM \in \langle \tau' M \rangle$ for some $\tau' \in G$ such that $\langle \tau' M \rangle \cap \langle \sigma M \rangle = M$ and $\langle \tau' M \rangle \cong C_{p^{m-1}}$. If $\langle \tau' \rangle \cap M = 1_G$, then $\Omega_1(G) \leq \langle \tau', M \rangle$ and so $\langle \sigma^{p^{m-1}} M \rangle = \Omega_1(G)/M \leq \langle \tau', M \rangle/M = \langle \tau' M \rangle$, a contradiction. Thus, $\langle \tau' \rangle \cap M \neq 1_G$, and hence $M \leq \langle \tau' \rangle$. This implies that $g \in \langle \tau' \rangle \cong C_{p^m}$, as required. \square

3.2 Proof of Theorem 1.1

We prove the following result first.

Lemma 3.5 *Let Γ be a connected weak metacirculant with order a power of an odd prime p . Then $\text{Aut}(\Gamma)$ contains a metacyclic p -subgroup G which is transitive on $V(\Gamma)$. Moreover, if $Z(G)$ is not cyclic, then G is regular on $V(\Gamma)$ and so Γ is a weak metacirculant Cayley graph.*

Proof. Since Γ is a weak metacirculant, $\text{Aut}(\Gamma)$ has a metacyclic subgroup X which is transitive on $V(\Gamma)$. Let G be a Sylow p -subgroup of X . Then G is metacyclic, and by [19, Theorem 3.4], G is also transitive on $V(\Gamma)$, proving the first statement in the lemma.

Since $p > 2$, we have $\Omega_1(G) \cong C_p \times C_p$ by Lemma 3.2. If $Z(G)$ is not cyclic, then $\Omega_1(G) \leq Z(G)$. For any $v \in V(\Gamma)$, if $G_v \neq 1_G$, then $G_v \cap \Omega_1(G) \neq 1_G$. However, $G_v \cap \Omega_1(G) \trianglelefteq G$. So $G_v \cap \Omega_1(G)$ fixes every vertex of Γ , a contradiction. Thus, $G_v = 1_G$, and so G is regular on $V(\Gamma)$. It follows that Γ is a Cayley graph of G . \square

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By [12, Lemma 2.2], each metacirculant has a vertex-transitive split metacyclic automorphism group. The necessity follows. It remains to prove the sufficiency.

Suppose that G is a split metacyclic vertex-transitive p -subgroup of $\text{Aut}(\Gamma)$. If G is regular on $V(\Gamma)$, then Γ is a Cayley graph of G . Since G is a split metacyclic group, Γ is a metacirculant graph, as required. In what follows we assume that G is not regular on $V(\Gamma)$. Then $Z(G)$ must be cyclic by Lemma 3.5.

Claim. G can be written as $\langle x \rangle : \langle y \rangle \cong C_{p^m} : C_{p^n}$ for some integers $m \geq n$.

In fact, by Lemma 3.1, we have

$$G = \langle a, b \mid a^{p^{r+s+u}} = 1_G, b^{p^{r+s+t}} = a^{p^{r+s}}, b^{-1}ab = a^{1+p^r} \rangle$$

for some nonnegative integers r, s, t, u with $r \geq 1, r \geq u$. A straightforward computation leads to the following observations:

- (i) $|G| = p^{2(r+s)+u+t}$, $\exp(G) = p^{r+s+t+u} = o(b)$;
- (ii) $G' = \langle a^{p^r} \rangle \cong C_{p^{s+u}}$ and G is p^{s+u} -abelian;
- (iii) $Z(G) = \langle a^{p^{s+u}}, b^{p^{s+u}} \rangle$.

Since $o(a^{p^{s+u}}) = p^r \leq o(b^{p^{s+u}}) = p^{r+t}$ and $Z(G)$ is cyclic, we then have $Z(G) = \langle b^{p^{s+u}} \rangle$. Consequently, $a^{p^{s+u}} \in \langle a \rangle \cap \langle b \rangle = \langle b^{p^{r+s+t}} \rangle = \langle a^{p^{r+s}} \rangle \cong C_{p^u}$ and so $r \leq u$. This together with $u \leq r$ implies $r = u$.

Since G is split, by Lemma 3.1, we have $stu = 0$. If $u = 0$, then $u = r = 0$, contradicting the assumption that $r \geq 1$. So $u > 0$. We then have

$$G = \begin{cases} \langle b \rangle : \langle ab^{-p^t} \rangle \cong C_{p^{2r+t}} : C_{p^r}, & \text{if } s = 0; \\ \langle a \rangle : \langle ba^{-1} \rangle \cong C_{p^{2r+s}} : C_{p^{r+s}}, & \text{if } t = 0 \end{cases}$$

as stated in the Claim.

By the Claim above, $G = \langle x \rangle : \langle y \rangle \cong C_{p^m} : C_{p^n}$ with $m \geq n$. Since G is transitive on $V(\Gamma)$, $\langle x \rangle$ acts semiregularly on $V(\Gamma)$. Assume that $\langle x \rangle$ has p^ℓ orbits for some $\ell \leq n$. For any $v \in V(\Gamma)$, let $G_v = \langle z \rangle$. Then $y^{p^\ell} \in \langle x \rangle : G_v$ and $|G_v| = p^{n-\ell}$. Moreover, $G_v \cap \langle x \rangle = 1_G$. It follows that $G_v \cong G_v \langle x \rangle / \langle x \rangle \leq G / \langle x \rangle \cong C_{p^n}$. By Lemma 3.4, there exists $y' \in G$ such that $\langle y' \rangle \cong C_{p^n}$ and $z \in \langle y' \rangle$. So G_v is a subgroup of $\langle y' \rangle$ of order $p^{n-\ell}$, and hence $\langle (y')^{p^\ell} \rangle = G_v$. Since $\langle x \rangle \cap G_v = 1_G$, we have $\langle x \rangle \cap \langle y' \rangle = 1_G$, and since $\langle y' \rangle \cong C_{p^n}$, we have $G = \langle x \rangle : \langle y' \rangle$. Then y' cyclically permutes the p^ℓ orbits of $\langle x \rangle$, and $(y')^{p^\ell} \in G_v$, implying that Γ is a metacirculant. \square

4 Smallest possible order and valency

The main result in this section is the following lemma, which asserts that for an odd prime p , if a weak metacirculant of order p^n that is a Cayley graph but not a weak metacirculant Cayley graph exists, then it has order at least p^4 and valency at least $2p + 2$. In the next two sections we will see that both p^4 and $2p + 2$ are attainable, as needed to establish the second statement in Theorem 1.2.

Lemma 4.1 *Let p be an odd prime. Let Γ be a weak metacirculant of order p^n for some integer $n \geq 1$. If Γ has valency less than $2p + 2$ or n is at most 3, then Γ must be a weak metacirculant Cayley graph.*

Proof. By Lemma 3.5, $\text{Aut}(\Gamma)$ has a metacyclic p -subgroup G which is transitive on $V(\Gamma)$. If G is regular on $V(\Gamma)$, then obviously Γ is a weak metacirculant Cayley graph. In what follows we assume that G is not regular on $V(\Gamma)$. Then G is non-abelian, and by Lemma 3.5, $Z(G)$ is cyclic. Since Γ has odd order p^n , its valency must be even. We are going to show that Γ is a circulant if it has valency less than $2p + 2$ or $1 \leq n \leq 3$.

If Γ has valency less than $2p$, then by [7, Lemm 2.4], G is regular, which contradicts our assumption. Suppose that Γ has valency $2p$. Since G is a metacyclic p -group with $p > 2$, we have $\Omega_1(G) \cong C_p \times C_p$ by Lemma 3.2. Since G is not regular on $V(\Gamma)$, we have $G_v > 1$ for $v \in V(\Gamma)$, and so $G_v \cap \Omega_1(G) > 1_G$. Consider the quotient graph $\Gamma_{\Omega_1(G)}$ of Γ relative to $\Omega_1(G)$. Each orbit of $\Omega_1(G)$ has length p , and the subgraph of Γ induced by any two adjacent orbits of $\Omega_1(G)$ is isomorphic to $K_{p,p}$. So $\Gamma_{\Omega_1(G)} \cong \text{Cyc}_{p^\ell}$ for some integer $\ell \geq 1$. Therefore, $\Gamma \cong \text{Cyc}_{p^\ell}[pK_1]$, which is a circulant.

Suppose that $1 \leq n \leq 3$. Since G is metacyclic, we may assume that $G = \langle \sigma, \tau \rangle$ with $\langle \sigma \rangle \trianglelefteq G$. Recall that G is transitive but not regular on $V(\Gamma)$ and G is non-abelian. Since $\langle \sigma \rangle \trianglelefteq G$, $\langle \sigma \rangle$ is semiregular on $V(\Gamma)$. In the following we will prove that $\langle \sigma \rangle$ is transitive on $V(\Gamma)$. Once this is achieved, it then follows that $\langle \sigma \rangle$ is regular on $V(\Gamma)$ and so Γ is a Cayley graph of $\langle \sigma \rangle$, as required.

Suppose to the contrary that $\langle \sigma \rangle$ is intransitive on $V(\Gamma)$. Since $n \leq 3$, we have $\langle \sigma \rangle \cong C_p$ or C_{p^2} . If $\langle \sigma \rangle \cong C_p$, then it is in the center of G , and so G is abelian, a contradiction. Thus

$\langle \sigma \rangle \cong C_{p^2}$. So $n = 3$ and τ induces an automorphism of $\langle \sigma \rangle$ of order p . It follows that $\sigma^\tau = \sigma^{kp+1}$ for some integer $1 \leq k \leq p-1$. This implies that $G' = \langle \sigma^p \rangle \cong \mathbb{Z}_p$. Since G is a 2-generator group, by elementary p -group theory (see, for example, [3, Lemma 65.2]), G is an inner abelian p -group and therefore $\Phi(G) = Z(G)$. (A group is inner abelian if it is non-abelian but all its proper subgroups are abelian.) Moreover, by [15] or [3, Lemma 65.1], we may assume that

$$G = \langle \sigma, \tau \mid \sigma^{p^2} = \tau^{p^s} = 1_G, \sigma^\tau = \sigma^{p+1} \rangle.$$

Take $v \in V(\Gamma)$. Since $\Phi(G) = Z(G)$, we have $G_v \cap \Phi(G) = 1_G$. Since G is not regular on $V(\Gamma)$, we have $G_v \neq 1_G$. Take $1_G \neq x \in G_v$. Then $x \notin \Phi(G)$. Since $x^p \in \Phi(G)$, we have $x^p = 1_G$. Since $x \notin \Phi(G) = Z(G)$, x commutes with at most one of σ and τ . If $x\sigma \neq \sigma x$, then $G = \langle \sigma, x \rangle$ because G is inner-abelian and so $|G| = p^3$. However, this is impossible as G is not regular on $V(\Gamma)$. If $x\sigma = \sigma x$, then $x\tau \neq \tau x$ and so $G = \langle \tau, x \rangle$. We may write $x = \sigma^i \tau^j$ for some integers i, j . Then $i \in \mathbb{Z}_{p^2}^*$ as $G = \langle \tau, x \rangle = \langle \tau, \sigma^i \tau^j \rangle$. Since $p > 2$, by Lemma 3.3, G is p -abelian, and hence $(\sigma^i \tau^j)^p = \sigma^{pi} \tau^{pj}$. It then follows that $1_G = x^p = \sigma^{pi} \tau^{pj}$. Thus, $\sigma^{pi} = 1_G$, but this is a contradiction as σ has order p^2 . \square

5 Multilayer generalized Petersen graphs

In this section we introduce a construction that can be viewed as a generalization of generalized Petersen graphs. In the next section, we will see that in a special case this construction gives rise to an infinite family of weak metacirculants of odd prime power order which are Cayley graphs but not weak metacirculant Cayley graphs, as needed to establish Theorem 1.2. Introduced in [18], generalized Petersen graphs are well studied, and they have been generalized in several ways in recent years (see, for example, [6, 16]). Our generalization is different from the existing ones.

Let $n \geq 3$ and $1 \leq t < n/2$. The *generalized Petersen graph* $P(n, t)$ is the graph with vertex set $\{x_i, y_i \mid i \in \mathbb{Z}_n\}$ and edge set the union of the *out edges* $\{\{x_i, x_{i+1}\} \mid i \in \mathbb{Z}_n\}$, the *inner edges* $\{\{y_i, y_{i+t}\} \mid i \in \mathbb{Z}_n\}$ and the *spokes* $\{\{x_i, y_i\} \mid i \in \mathbb{Z}_n\}$. It is evident that $P(n, t)$ has an automorphism $\alpha = (x_0 x_1 x_2 \dots x_{n-1})(y_0 y_1 y_2 \dots y_{n-1})$ and that $H = \langle \alpha \rangle$ is semiregular on the vertex set of $P(n, t)$ with two orbits, namely $X = \{x_i \mid i \in \mathbb{Z}_n\}$ and $Y = \{y_i \mid i \in \mathbb{Z}_n\}$. The subgraph of $P(n, t)$ induced by X is an n -cycle while the subgraph of $P(n, t)$ induced by Y is the union of some vertex-disjoint cycles.

We now generalize generalized Petersen graphs in such a way that the cyclic semiregular subgroup H has $m \geq 2$ orbits on the vertex set and that the subgraph induced on each orbit of H is a lexicographic product of the union of some cycles of equal length and an empty graph.

Definition 5.1 Let m, n, s and t be positive integers such that $m \geq 3, n \geq 2, s \mid m$ and $1 \leq t < \frac{m}{2}$. Let $H = \langle h \rangle \cong C_m$. For each $i \in \mathbb{Z}_n$, let

$$\begin{aligned} V_i &= \{(h^j, i) \mid j \in \mathbb{Z}_m\}, \\ E_i &= \{\{(h^j, i), (h^{j+ks+t^i}, i)\}, \{(h^j, i), (h^{j-ks-t^i}, i)\} \mid k \in \mathbb{Z}_{\frac{m}{s}}, j \in \mathbb{Z}_m\}, \\ E_{i,i+1} &= \{\{(h^j, i), (h^j, i+1)\} \mid j \in \mathbb{Z}_m\}, \end{aligned}$$

where subscripts are modulo n . Define the graph $\Gamma = \text{MP}_{m,n,s,t}$ by

$$V(\Gamma) = \cup_{i \in \mathbb{Z}_n} V_i, \quad E(\Gamma) = \cup_{i \in \mathbb{Z}_n} (E_i \cup E_{i,i+1})$$

and call it the *multilayer generalized Petersen graph* with parameters (m, n, s, t) .

It can be verified that $\text{MP}_{m,2,m,t}$ is exactly the generalized Petersen graph $P(m, t)$.
For each $g \in H$, define the permutation $R(g)$ on the vertices of $\Gamma = \text{MP}_{m,n,s,t}$ by

$$(h^j, i)^{R(g)} = (h^j g, i), \text{ for } i \in \mathbb{Z}_n, j \in \mathbb{Z}_m.$$

Let $R(H) = \{R(g) \mid g \in H\}$. A simple computation shows that $R(H)$ is a semiregular subgroup of $\text{Aut}(\Gamma)$ isomorphic to H whose orbits on $V(\Gamma)$ are V_i , $i \in \mathbb{Z}_n$. Moreover, for each $i \in \mathbb{Z}_n$ the edges between V_i and V_{i+1} form a perfect matching, the subgraph $\Gamma[V_0]$ of Γ induced by V_0 is isomorphic to the lexicographic product $\text{Cyc}_s[\frac{m}{s}K_1]$, and for $i \in \mathbb{Z}_n \setminus \{0\}$ the subgraph $\Gamma[V_i]$ of Γ induced by V_i is isomorphic to the lexicographic product of the union of some ℓ -cycles and $\frac{m}{s}K_1$, where $\ell \mid s$. Hence if $n > 2$, then Γ has valency $2(\frac{m}{s} + 1)$, while if $n = 2$, then it has valency $\frac{2m}{s} + 1$.

Lemma 5.2 *Let $\Gamma = \text{MP}_{m,n,s,t}$. If $m/s > 1$ and $t \in \mathbb{Z}_m^*$, then each V_i is a block of imprimitivity of $\text{Aut}(\Gamma)$ on $V(\Gamma)$.*

Proof. Since $t \in \mathbb{Z}_m^*$, for each $i \in \mathbb{Z}_n$, $\Gamma[V_i] \cong \text{Cyc}_s[\frac{m}{s}K_1]$. It suffices to prove that V_0 is a block of imprimitivity of $\text{Aut}(\Gamma)$ on $V(\Gamma)$. Suppose that $V_0^g \cap V_0 \neq \emptyset$ for some $g \in \text{Aut}(\Gamma)$. Take $v_0 = u_0^g \in V_0^g \cap V_0$. Suppose that v_0 has a neighbour, say x_0^g in V_0^g but not in V_0 . Since the edges between V_i and V_{i+1} are independent for any $i \in \mathbb{Z}_n$, v_0 is the only neighbour of x_0^g in V_0 .

Since $\Gamma[V_0] \cong \text{Cyc}_s[\frac{m}{s}K_1]$, we may assume that all the vertices in $\{u_0, u_1, \dots, u_{\frac{m}{s}-1}^g\}$ have the same neighbourhood in $\Gamma[V_0]$. This implies that all vertices in the set $U = \{v_0 = u_0^g, u_1^g, \dots, u_{\frac{m}{s}-1}^g\}$ have the same neighbourhood in $\Gamma[V_0^g]$. In particular, x_0^g is adjacent to each vertex in U . Since v_0 is the only neighbour of x_0^g in V_0 , we have $U \cap V_0 = \{v_0\}$. Note that v_0 has only two neighbours outside of V_0 , and Γ has valency $2(\frac{m}{s} + 1)$. Hence $|\Gamma(v_0) \cap V_0 \cap V_0^g| \geq 2(\frac{m}{s} - 1) \geq 2$ as $\frac{m}{s} \geq 2$. Observe that each vertex in U is adjacent to all vertices in $\Gamma(v_0) \cap V_0 \cap V_0^g$. However, since $u_1^g \notin V_0$, u_1^g has at most one neighbour in V_0 , which is a contradiction. Thus all neighbours of v_0 in V_0^g are contained in V_0 . By the arbitrariness of v_0 , we have $V_0^g \subseteq V_0$ and so $V_0^g = V_0$, completing the proof. \square

6 Proof of Theorem 1.2

The purpose of this section is to prove the following result, which together with Lemma 4.1 implies Theorem 1.2.

Theorem 6.1 *Let p be an odd prime, m and n be integers with $m \geq n + 2 \geq 3$, and λ be an element of $\mathbb{Z}_{p^m}^*$ with order p^{n+1} . Then $\text{MP}_{p^m, p^n, p^{m-1}, \lambda}$ is a weak metacirculant which is a Cayley graph but not a weak metacirculant Cayley graph.*

In the rest of this section, we always let p, m, n and λ be as in Theorem 6.1, and $H = \langle h \rangle \cong C_m$ be as in Definition 5.1. By Definition 5.1, $\text{MP}_{p^m, p^n, p^{m-1}, \lambda}$ has vertex set $\cup_{i \in \mathbb{Z}_{p^n}} V_i$ and edge set $\cup_{i \in \mathbb{Z}_{p^n}} (E_i \cup E_{i,i+1})$, where for each $i \in \mathbb{Z}_{p^n}$,

$$\begin{aligned} V_i &= \{(h^j, i) \mid j \in \mathbb{Z}_{p^m}\}, \\ E_i &= \{(h^j, i), (h^{j+kp^{m-1}+\lambda^i}, i)\}, \{(h^j, i), (h^{j-kp^{m-1}-\lambda^i}, i)\} \mid k \in \mathbb{Z}_p, j \in \mathbb{Z}_{p^m}\}, \\ E_{i,i+1} &= \{(h^j, i), (h^j, i+1)\} \mid j \in \mathbb{Z}_{p^m}\}. \end{aligned}$$

The proof of Theorem 6.1 consists of the following three lemmas.

Lemma 6.2 *The graph $\text{MP}_{p^m, p^n, p^{m-1}, \lambda}$ is a metacirculant. Moreover, $\text{Aut}(\text{MP}_{p^m, p^n, p^{m-1}, \lambda})$ is transitive on the set of those arcs of $\text{MP}_{p^m, p^n, p^{m-1}, \lambda}$ whose underlying edges are in $\cup_{i \in \mathbb{Z}_{p^n}} E_i$.*

Proof. Denote $\Gamma = \text{MP}_{p^m, p^n, p^{m-1}, \lambda}$. Recall that for $g \in H$, $R(g)$ is the permutation on the vertices of Γ defined by:

$$(h^j, i)^{R(g)} = (h^j g, i), \text{ for } i \in \mathbb{Z}_{p^n}, j \in \mathbb{Z}_{p^m}.$$

Recall also that $R(H) = \{R(g) \mid g \in H\}$ is a semiregular subgroup of $\text{Aut}(\Gamma)$ isomorphic to H whose orbits on $V(\Gamma)$ are $V_i, i \in \mathbb{Z}_{p^n}$.

Let α be the automorphism of H such that $h^\alpha = h^\lambda$. Define a permutation σ_α on the vertices of Γ by

$$(h^j, i)^{\sigma_\alpha} = ((h^j)^\alpha, i + 1), \text{ for } i \in \mathbb{Z}_{p^n}, j \in \mathbb{Z}_{p^m}.$$

For each $i \in \mathbb{Z}_{p^n}$, it is easy to see that $E_{i, i+1}^{\sigma_\alpha} = E_{i+1, i+2}$. Furthermore,

$$\begin{aligned} \{(h^j, i), (h^{j+kp^{m-1}+\lambda^i}, i)\}^{\sigma_\alpha} &= \{(h^{j\lambda}, i+1), (h^{j\lambda+k\lambda p^{m-1}+\lambda^{i+1}}, i+1)\} \in E_{i+1}, \\ \{(h^j, i), (h^{j-kp^{m-1}-\lambda^i}, i)\}^{\sigma_\alpha} &= \{(h^{j\lambda}, i+1), (h^{j\lambda-k\lambda p^{m-1}-\lambda^{i+1}}, i+1)\} \in E_{i+1}, \end{aligned}$$

and so $E_i^{\sigma_\alpha} = E_{i+1}$ for each $i \in \mathbb{Z}_{p^n}$. This implies that σ_α preserves the adjacency relation of Γ , and so $\sigma_\alpha \in \text{Aut}(\Gamma)$.

For any $(h^j, i) \in V(\Gamma)$, we have $(h^j, i)^{R(h)\sigma_\alpha} = (h^{(j+1)\lambda}, i+1) = (h^j, i)^{\sigma_\alpha R(h^\lambda)}$. It follows that $R(h)\sigma_\alpha = R(h^\lambda) = R(h)^\lambda$, and so $\langle R(h), \sigma_\alpha \rangle$ is metacyclic. Clearly, $\langle R(h), \sigma_\alpha \rangle$ is transitive on $V(\Gamma)$, and $((1, 0), (1, 1), \dots, (1, p^n - 1))$ is a cycle of σ_α (as a permutation on $V(\Gamma)$). So Γ is a metacirculant.

The subgraph of Γ induced by V_0 is $\Gamma_0 = (V_0, E_0)$. It can be verified that $\langle R(h) \rangle : \langle \sigma_\alpha^{p^n} \rangle \cong C_{p^m} : C_p$ acts transitively on E_0 . Let β be the automorphism of H inverting every element of H . Let σ_β be a permutation of $V(\Gamma)$ such that

$$(h^j, i)^{\sigma_\beta} = ((h^j)^\beta, i), \text{ for } i \in \mathbb{Z}_{p^n}, j \in \mathbb{Z}_{p^m}.$$

One can verify that $\sigma_\beta \in \text{Aut}(\Gamma)$ and σ_β fixes each V_i setwise. Furthermore, $R(h^{-1})\sigma_\beta$ takes the arc $((1, 0), (h, 0))$ to its inverse arc $((h, 0), (1, 0))$. This implies that $\langle R(h), \sigma_\alpha^{p^n}, \sigma_\beta \rangle$ is transitive on the set of arcs of Γ_0 . Since σ_α cyclically permutes V_i 's, it follows that $\text{Aut}(\Gamma)$ is transitive on those arcs of Γ whose underlying edges are in $\cup_{i \in \mathbb{Z}_{p^n}} E_i$. \square

Let

$$\mathcal{G} = \langle x, y, z \mid x^{p^{m-1}} = y^{p^n} = z^p = 1, y^{-1}xy = x^\lambda, [z, x] = [z, y] = 1 \rangle.$$

It is easily seen that $\langle x^{p^{m-2}}, y^{p^{n-1}}, z \rangle \cong C_p^3$, and so \mathcal{G} is a non-metacyclic group.

Lemma 6.3 $\text{MP}_{p^m, p^n, p^{m-1}, \lambda} \cong \text{Cay}(\mathcal{G}, S \cup S^{-1})$, where

$$S = \{x, xz, xz^2, \dots, xz^{(p-1)}, y\}.$$

Proof. Denote $\Gamma = \text{MP}_{p^m, p^n, p^{m-1}, \lambda}$ and $\Sigma' = \text{Cay}(\mathcal{G}, S \cup S^{-1})$. Define

$$f : y^i x^j z^k \mapsto (h^{kp^{m-1}+j}, i), \text{ for } i \in \mathbb{Z}_{p^n}, j \in \mathbb{Z}_{p^{m-1}}, k \in \mathbb{Z}_p.$$

It can be verified that f is a bijection from $V(\Sigma')$ to $V(\Gamma)$. The neighbourhood of $y^i x^j z^k$ in Σ' is

$$\{y^i x^{j \pm \lambda^i} z^{k \pm l} \mid l \in \mathbb{Z}_p\} \cup \{y^{i+1} x^j z^k, y^{i-1} x^j z^k\}.$$

The image of this set under f is

$$\{(h^{kp^{m-1}+j \pm (lp^{m-1} + \lambda^i)}, i) \mid l \in \mathbb{Z}_p\} \cup \{(h^{kp^{m-1}+j}, i+1), (h^{kp^{m-1}+j}, i-1)\},$$

which is exactly the neighbourhood of $f(y^i x^j z^k) = (h^{kp^{m-1}+j}, i)$ in Γ . Therefore, f is an isomorphism from Σ' to Γ . \square

Lemma 6.4 *The graph $\text{MP}_{p^m, p^n, p^{m-1}, \lambda}$ is not a weak metacirculant Cayley graph.*

Proof. Denote $\Gamma = \text{MP}_{p^m, p^n, p^{m-1}, \lambda}$ and $A = \text{Aut}(\Gamma)$. We first prove the following claim.

Claim. For any $i \in \mathbb{Z}_{p^n}$ and $j \in \mathbb{Z}_{p^m}$, if $j \not\equiv 0 \pmod{p^{m-1}}$, then the distance between $(1, i)$ and (h^j, i) in $\Gamma[V_i]$ is $\min\{t, p^{m-1} - t\}$, where $t\lambda^i \equiv j \pmod{p^{m-1}}$ and $0 \leq t \leq p^{m-1} - 1$.

Given $i \in \mathbb{Z}_{p^n}$ and $0 \leq \ell \leq p^{m-1} - 1$, let

$$V_{i\ell} = \{(h^{kp^{m-1}+\ell}, i) \mid k \in \mathbb{Z}_p\}.$$

Then

$$\mathcal{B} = \{V_{i\ell} \mid 0 \leq \ell \leq p^{m-1} - 1\}$$

is a partition of V_i . Moreover, each $V_{i\ell}$ is an independent set of Γ , and the subgraph of Γ induced by $V_{i\ell} \cup V_{i(\ell+\lambda^i)}$ is isomorphic to $K_{p,p}$. Since $\Gamma[V_i] \cong \text{Cyc}_{p^{m-1}}[pK_1]$, the quotient graph Y of $\Gamma[V_i]$ relative to \mathcal{B} is a cycle of length p^{m-1} .

For any $j \in \mathbb{Z}_{p^m}$, if $j \not\equiv 0 \pmod{p^{m-1}}$, then there exists $1 \leq \ell \leq p^{m-1} - 1$ such that $j \equiv \ell \pmod{p^{m-1}}$, and moreover, there exists $0 \leq t \leq p^{m-1} - 1$ such that $t\lambda^i \equiv \ell \pmod{p^{m-1}}$. The distance d between $(1, i)$ and (h^j, i) in $\Gamma[V_i]$ is just the distance between V_{i0} and $V_{i\ell} = V_{i(t\lambda^i)}$ in the quotient graph Y . It follows that $d = \min\{t, p^{m-1} - t\}$, completing the proof of the Claim.

By Lemma 5.2, for each $i \in \mathbb{Z}_{p^n}$, V_i is a block of imprimitivity of A on $V(\Gamma)$. It follows that $\cup_{i \in \mathbb{Z}_{p^n}} \Gamma[V_i]$ is a $2p$ -factor of Γ which is invariant under the action of A . So A preserves the set

$$F = \cup_{i \in \mathbb{Z}_{p^n}} E_i.$$

Let $\Gamma' = \Gamma - F$. Then $A \leq \text{Aut}(\Gamma')$, and for each $j \in \mathbb{Z}_{p^m}$, $\Gamma[B_j]$ is a subgraph of Γ' isomorphic to Cyc_{p^n} , where

$$B_j = \{(h^j, 0), (h^j, 1), (h^j, 2), \dots, (h^j, p^n - 1)\}.$$

So $\Gamma' \cong p^m \text{Cyc}_{p^n}$ is the union of p^m vertex-disjoint cycles of length p^n .

Suppose that Γ is a Cayley graph of a metacyclic p -group G . Say, $\Gamma \cong \Lambda = \text{Cay}(G, S)$ for an inverse-closed subset S of $G \setminus \{1_G\}$ that generates G . Recall that $\Gamma' \cong p^m \text{Cyc}_{p^n}$ is a 2-factor of Γ invariant under A , $\Gamma - E(\Gamma') \cong p^n \text{Cyc}_{p^{m-1}}[pK_1]$, and for any two components of $\Gamma - E(\Gamma')$, either there is no edge connecting them in Γ or the edges between them form a perfect matching. Since $\Gamma \cong \Lambda$, Λ also has a 2-factor $\Lambda' \cong p^m \text{Cyc}_{p^n}$ invariant under $\text{Aut}(\Lambda)$ such that $\Lambda - E(\Lambda') \cong p^n \text{Cyc}_{p^{m-1}}[pK_1]$ and for any two components of $\Lambda - E(\Lambda')$, either there is no edge connecting them in Λ or the edges between them form a perfect matching. So there exists $x \in S$ such that $\{1_G, x\}$ is an edge of Λ' . Since $\text{Aut}(\Lambda) \leq \text{Aut}(\Lambda')$, $\{1_G, x\}^{R(x^\ell)} = \{x^\ell, x^{\ell+1}\} \in E(\Lambda')$ for any $0 \leq \ell \leq o(x)$, which implies that $(1_G, x, x^2, \dots, x^{o(x)-1})$ is cycle of Λ' . Since $\Lambda' \cong p^m \text{Cyc}_{p^n}$, we have $o(x) = p^n$. Clearly, $x^{-1} \in S$ and $\text{Cay}(G, \{x, x^{-1}\}) \cong p^m \text{Cyc}_{p^n}$. Recall that G acts faithfully on $V(\Lambda)$ by right multiplication, and this action induces a regular subgroup $R(G)$ of

$\text{Aut}(\Lambda)$, which will be identified with G in the sequel. Let P be a Sylow p -subgroup of $\text{Aut}(\Lambda)$ such that $G \leq P$. From the proof of Lemma 6.2, we see that G is a proper subgroup of P , and so $N_P(G) > G$. It follows that $p \mid |\text{Aut}(G, S)|$. Let $\beta \in \text{Aut}(G, S)$ be of order p . Then β preserves $\text{Cay}(G, \{x, x^{-1}\}) \cong p^m \text{Cyc}_{p^n}$, and so β must fix x . Consequently, $p \mid |\text{Aut}(G, S - \{x, x^{-1}\})|$, and hence $S = \{x, x^{-1}\} \cup \{y^{\beta^i}, (y^{-1})^{\beta^i} \mid i \in \mathbb{Z}_p\}$ for some $y \in S - \{x, x^{-1}\}$.

Since $\Lambda - E(\Lambda') \cong p^n \text{Cyc}_{p^{m-1}}[pK_1]$, we have $\Lambda - E(\Lambda') = \text{Cay}(G, S - \{x, x^{-1}\})$. Let $T = \langle S - \{x, x^{-1}\} \rangle$. Then $\text{Cay}(T, S - \{x, x^{-1}\})$ is a subgraph of Λ isomorphic to $\text{Cyc}_{p^{m-1}}[pK_1]$. So T has order p^m . Then $G = \cup_{i \in \mathbb{Z}_{p^n}} Tx^i$. Recall that the p^m edges of Λ between T and Tx are independent. So the quotient graph of Λ relative to $\{Tx^i \mid i \in \mathbb{Z}_{p^n}\}$ is a cycle of length p^n . Let K be the kernel of G acting on $\{Tx^i \mid i \in \mathbb{Z}_{p^n}\}$. Then $G/K \leq D_{2p^n}$ and so $G/K \cong \mathbb{Z}_{p^n}$ as G is a p -group. This implies that $T = K$. We claim that T is cyclic. Suppose on the contrary that T is non-cyclic. Then G is non-cyclic. Since G is metacyclic and $p > 2$, by Lemma 3.2 the subgroup $\Omega_1(G)$ of G generated by the elements of order p is an elementary abelian group of order p^2 . Hence $\langle x \rangle \cap \Omega_1(G) \neq 1_G$ and $\Omega_1(G) \leq T$. Consequently, $T \cap \langle x \rangle \neq 1_G$. Note that $\langle x \rangle$ acts transitively on the p^n components of $\text{Cay}(G, S - \{x, x^{-1}\})$, and T is the stabilizer of the block T in G . Hence $G = T\langle x \rangle$. Since $|T| = p^m$ and $o(x) = p^n$, from $|G| = p^{m+n}$ it follows that $\langle x \rangle \cap T = 1_G$, a contradiction. Thus, T is a normal cyclic subgroup of G of order p^m , and moreover, $G = T : \langle x \rangle$. Therefore, $T = \langle y \rangle$ and $S - \{x, x^{-1}\} = \{y^{kp^{m-1}+1}, y^{-kp^{m-1}-1} \mid k \in \mathbb{Z}_p\}$.

Assume that $y^x = y^{\lambda'}$ for some $\lambda' \in \mathbb{Z}_{p^m}^*$. Since x has order p^n , λ' has order at most p^n . Recall that the edges of Λ between T and Tx^{-1} form a perfect matching. Note that $1_G \sim x^{-1}$ and $y \sim x^{-1}y$. Since $x^{-1}yx = y^{\lambda'}$, we have $x^{-1}y = y^{\lambda'}x^{-1}$. We now consider the distance d' between x^{-1} and $x^{-1}y = y^{\lambda'}x^{-1}$ in the subgraph induced by Tx^{-1} . Indeed, d' is just the distance between 1_G and $y^{\lambda'}$ in the subgraph induced by T . Observe that the subgraph induced by T is the Cayley graph

$$\text{Cay}(T, \{y^{kp^{m-1}+1}, y^{-kp^{m-1}-1} \mid k \in \mathbb{Z}_p\}),$$

which is isomorphic to $\text{Cyc}_{p^{m-1}}[pK_1]$. Let $B = \langle y^{p^{m-1}} \rangle$. Then

$$T = \cup_{\ell \in \mathbb{Z}_{p^{m-1}}} By^\ell.$$

It is clear that each coset By^ℓ is an independent set of $\Lambda[T]$, and the subgraph of Λ induced by $By^\ell \cup By^{\ell+1}$ is isomorphic to $K_{p,p}$. Note that $By^{\lambda'} = By^t$, where $\lambda' \equiv t' \pmod{p^{m-1}}$ and $1 \leq t' \leq p^{m-1} - 1$. Hence $d' = \min\{t', p^{m-1} - t'\}$.

Suppose that f is an isomorphism from Λ to Γ . Since Γ is vertex-transitive, we may assume that f maps 1_G to $(1, 0)$. By Lemma 6.2, the arcs in $\Gamma[V_0]$ are equivalent under A . So we may further assume that f takes the arc $(1_G, y)$ of Λ to the arc $((1, 0), (h, 0))$ of Γ . Clearly, f maps Tx^{-1} to V_1 or V_{p^n-1} .

If f maps Tx^{-1} to V_1 , then since the edges between T and Tx form a perfect matching, f maps x^{-1} and $x^{-1}y (= y^{\lambda'}x^{-1})$ to $(1, 1)$ and $(h, 1)$, respectively. By the Claim above, the distance between $(1, 1)$ and $(h, 1)$ in $\Gamma[V_1]$ is $\min\{t, p^{m-1} - t\}$, where $t\lambda \equiv 1 \pmod{p^{m-1}}$ and $1 \leq t \leq p^{m-1} - 1$. Hence $\min\{t', p^{m-1} - t'\} = \min\{t, p^{m-1} - t\}$. It follows that $\lambda^{-1} \equiv \lambda' \pmod{p^{m-1}}$ because λ and λ' have odd orders, where λ^{-1} is the inverse of λ in \mathbb{Z}_n^* . Consequently, $\lambda^{-1} = kp^{m-1} + \lambda'$ for some $k \in \mathbb{Z}_{p^m}$. Hence $\lambda^{-p} \equiv (\lambda')^p \pmod{p^m}$. This implies that the order of $\lambda^{-1} \in \mathbb{Z}_{p^m}^*$ is at most p^n , a contradiction.

Similarly, if f maps Tx^{-1} to V_{p^n-1} , then f maps x^{-1} and $x^{-1}y (= y^{\lambda'}x^{-1})$ to $(1, p^n - 1)$ and $(h, p^n - 1)$, respectively. Again by the Claim, the distance between $(1, 1)$ and $(h, p^n - 1)$ is $\min\{t, p^{m-1} - t\}$, where $t\lambda^{p^n-1} \equiv 1 \pmod{p^{m-1}}$ and $1 \leq t \leq p^{m-1} - 1$. Hence $\min\{t', p^{m-1} - t'\} =$

$\min\{t, p^{m-1} - t\}$. It follows that $\lambda^{1-p^n} \equiv \lambda' \pmod{p^{m-1}}$ because λ and λ' have odd orders. Consequently, $\lambda^{1-p^n} = kp^{m-1} + \lambda'$ for some $k \in \mathbb{Z}_{p^m}$. Hence $(\lambda^{1-p^n})^p \equiv (\lambda')^p \pmod{p^m}$. This implies that the order of $\lambda^{1-p^n} \in \mathbb{Z}_{p^m}^*$ is at most p^n . However, λ^{1-p^n} and λ have the same order which is assumed to be p^{n+1} , a contradiction. \square

So far we have proved Theorem 6.1. As mentioned earlier, Theorem 1.2 follows from Theorem 6.1 and Lemma 4.1.

7 Proof of Theorem 1.3

Let p be an odd prime and Γ a connected metacirculant graph of order p^4 and valency $2p + 2$. By Theorem 1.1, Γ has a split metacyclic vertex-transitive group of automorphisms, say G . We may further assume that G is a p -group. If $Z(G)$ is non-cyclic, then G is regular on $V(\Gamma)$ by Lemma 3.5, and so Γ is a metacirculant Cayley graph. If G is abelian, then again G is regular on $V(\Gamma)$ and Γ is a metacirculant Cayley graph.

In what follows, we assume that $Z(G)$ is cyclic and G is non-abelian. From the proof of Theorem 1.1, we may assume that $G = \langle x \rangle : \langle y \rangle \cong C_{p^m} : C_{p^n}$ for some $m \geq n$ and $G_v \leq \langle y \rangle$ for some $v \in V(\Gamma)$. Since G_v is core-free in G , every non-identity element of $\langle y \rangle$ induces a non-trivial automorphism of $\langle x \rangle$ by conjugation. Since $\text{Aut}(C_{p^m}) \cong C_{p^{m-1}} : C_{p-1}$, it follows that $n \leq m - 1$. Since $\langle x \rangle \trianglelefteq G$, the transitivity of G on $V(\Gamma)$ implies that $\langle x \rangle$ acts semiregularly on $V(\Gamma)$. Since $|V(\Gamma)| = p^4$, we have $m \leq 4$ and $m + n \geq 4$. If $m = 4$, then $\langle x \rangle$ acts regularly on $V(\Gamma)$ and so Γ is a metacirculant Cayley graph. If $m < 4$ and $m + n = 4$, then G acts regularly on $V(\Gamma)$ and hence Γ is a metacirculant Cayley graph.

Assume $m < 4 < m + n$. Then the only possibility is $(m, n) = (3, 2)$, which implies that $G_v = \langle y^p \rangle \cong C_p$ and $y^{-1}xy = x^{kp+1}$ for some $k \in \mathbb{Z}_p^*$. Consequently, $G' = \langle x^p \rangle \cong C_{p^2}$. Since $G = \langle x \rangle : \langle y \rangle \cong C_{p^3} : C_{p^2}$ and $p > 2$, from Lemma 3.2 it follows that $\Omega_1(G) = \langle x^{p^2} \rangle \times \langle y^p \rangle \cong C_p \times C_p$. By the N/C theorem, we have $G/C_G(\Omega_1(G)) \leq \text{Aut}(\Omega_1(G)) \cong \text{GL}(2, p)$. Since $|\text{GL}(2, p)| = (p^2 - p)(p^2 - 1)$ and G is a p -group, we have $G/C_G(\Omega_1(G)) = 1$ or $G/C_G(\Omega_1(G)) \cong C_p$. The former cannot happen, for otherwise $\Omega_1(G)$ is contained in $Z(G)$ and hence $G_v \trianglelefteq G$, a contradiction. So $G/C_G(\Omega_1(G)) \cong C_p$. Since $G_v \leq \Omega_1(G)$, $C_G(\Omega_1(G)) \leq C_G(G_v) \leq N_G(G_v)$ and hence $C_G(\Omega_1(G)) = C_G(G_v) = N_G(G_v)$ as G_v is non-normal in G .

By Proposition 2.1, Γ is isomorphic to the coset graph $\Gamma' = \text{Cos}(G, G_v, G_vSG_v)$, where S consists of the elements of G each of which maps v to one of its neighbours. We will simply identify Γ with Γ' in the remainder of the proof. Since Γ is connected, we have $G = \langle G_vSG_v \rangle$, which implies that there exists $d \in S \setminus C_G(G_v)$. Then $d = x^i y^j$ for some $i \in \mathbb{Z}_{p^3}^*$ and $j \in \mathbb{Z}_{p^2}$, and furthermore, $G_v^d \neq G_v$ and so $G_v^d \cap G_v = 1_G$ as $G_v \cong C_p$. Consequently, $|G_v d G_v|/|G_v| = p$. If $d^{-1} \in G_v d G_v$, then the subgraph $\text{Cos}(\langle G_v d G_v \rangle, G_v, G_v d G_v)$ of Γ would have odd order and odd valency p , but this cannot happen. Thus $d^{-1} \notin G_v d G_v$ and $|G_v \{d, d^{-1}\} G_v|/|G_v| = 2p$ by Lemma 2.2. We may assume that $D = G_v \{d, d^{-1}\} G_v \cup G_v \{c, c^{-1}\} G_v$ for some $c \in G$. Since Γ has valency $2p + 2$, we have $|G_v \{c, c^{-1}\} G_v|/|G_v| = 2$. It follows that c normalizes G_v and so $c \in N_G(G_v) = C_G(G_v)$.

Let $M = \langle d, G_v \rangle$. As $d \notin C_G(G_v) = \langle x^p \rangle : \langle y \rangle$, $o(d) = p^3$ and so $\langle d^{p^2} \rangle = \langle x^{p^2} \rangle$. It follows that $\Omega_1(G) \leq M$ and $M/\Omega_1(G) = \langle d\Omega_1(G) \rangle \cong C_{p^2}$. Consequently, $|M| = p^4$ and so $M \trianglelefteq G$.

Let $\Gamma_1 = \text{Cos}(M, G_v, G_v \{d, d^{-1}\} G_v)$. Then Γ_1 has order p^3 and valency $2p$, and M is a vertex- and edge-transitive group of automorphisms of Γ_1 (see Lemma 2.2). Recall that $\Omega_1(G) = \langle d^{p^2} \rangle \times G_v \cong C_p \times C_p$. The quotient graph $(\Gamma_1)_{\Omega_1(G)}$ of Γ_1 relative to $\Omega_1(G)$ has p^2 vertices, $\Omega_1(G)$ is the kernel of M acting on $V((\Gamma_1)_{\Omega_1(G)})$, and $M/\Omega_1(G)$ is edge-transitive

on $(\Gamma_1)_{\Omega_1(G)}$. Since $G_v \leq \Omega_1(G)$, the subgraph induced by some pair of adjacent orbits of $\Omega_1(G)$ is isomorphic to $K_{p,p}$. Since $M/\Omega_1(G)$ is edge-transitive on $(\Gamma_1)_{\Omega_1(G)}$, it follows that the subgraph induced by any two adjacent orbits of $\Omega_1(G)$ is isomorphic to $K_{p,p}$. Consequently, $(\Gamma_1)_{\Omega_1(G)} \cong \text{Cyc}_{p^2}$ and $\Gamma_1 \cong \text{Cyc}_{p^2}[pK_1]$. Since G acts transitively on $V(\Gamma)$, this implies that the subgraph of Γ induced by each orbit of M is isomorphic to $\text{Cyc}_{p^2}[pK_1]$.

Consider the quotient graph Γ_M of Γ relative to M . Since $M \trianglelefteq G$ and the subgraph of Γ induced by each orbit of M is isomorphic to $\text{Cyc}_{p^2}[pK_1]$, we have $\Gamma_M \cong \text{Cyc}_p$. So we may assume $V(\Gamma_M) = \{B_i \mid i \in \mathbb{Z}_p\}$ such that $B_i \sim B_{i+1}$ for $i \in \mathbb{Z}_p$. Then $\Gamma[B_i \cup B_{i+1}] \cong p^3K_2$ for each $i \in \mathbb{Z}_p$. By a similar argument as in the proof of Lemma 5.2, one can show that $V(\Gamma_M)$ is a system of blocks of imprimitivity of $\text{Aut}(\Gamma)$.

Assume that $c^p \notin G_v$. Since $c \in C_G(\Omega_1(G)) = N_G(G_v) \cong C_{p^2} : C_{p^2}$, c has order p^2 and so $(G_v, G_v c, \dots, G_v c^{p^2-1})$ is a cycle of Γ of length p^2 . Furthermore, $\langle c^p \rangle \times G_v = \Omega_1(G)$, so $\{G_v, G_v c^p, G_v c^{2p}, \dots, G_v c^{(p-1)p}\}$ is an orbit of $\Omega_1(G)$, and the vertices in this orbit have the same neighbourhood in Γ_1 .

If Γ is not a Cayley graph, then case (b) in Theorem 1.3 holds. In the sequel we assume that Γ is a Cayley graph of a group N , say $\Gamma = \text{Cay}(N, S)$. Let K be the kernel of N acting on $V(\Gamma_M)$. Then $N/K \cong C_p$. Recall that $\Gamma[B_0] \cong \text{Cyc}_{p^2}[pK_1]$. We call a subset of B_0 a *part* of B_0 if it is maximal with respect to the property that all vertices in the set have the same neighbourhood in $\Gamma[B_0]$. Since $\Gamma[B_0] \cong \text{Cyc}_{p^2}[pK_1]$, B_0 has p^2 parts, and each of them is a block of imprimitivity of $\text{Aut}(\Gamma[B_0])$ on B_0 . Let L be the kernel of K acting on the p^2 parts of $\Gamma[B_i]$. Then $K/L \cong C_{p^2}$ and $L \cong C_p$. This implies that $K \cong C_{p^3}$ or $K \cong C_{p^2} \times C_p$. In the former case, N is a metacyclic group, and so Γ is a metacirculant Cayley graph. Suppose $K \cong C_{p^2} \times C_p$ in what follows.

Since $V(\Gamma_M)$ is a system of blocks of imprimitivity of $\text{Aut}(\Gamma)$ on $V(\Gamma)$, we have $\text{Aut}(\Gamma) \leq \text{Aut}(\Gamma^*)$, where Γ^* is obtained from Γ by deleting the edges contained in each $\Gamma[B_0]$. Since $(G_v, G_v c, \dots, G_v c^{p^2-1})$ is a cycle of length p , we have $\Gamma^* \cong p^2 \text{Cyc}_{p^2}$. Since $\Gamma = \text{Cay}(N, S)$, we may relabel the vertices of Γ by the elements of N . Then S contains an element $g \in N \setminus K$ such that $o(g) = p^2$ and $(1_N, g, g^2, \dots, g^{p^2-1})$ is a cycle of Γ corresponding to $(G_v, G_v c, \dots, G_v c^{p^2-1})$. Recall that $\{G_v, G_v c^p, G_v c^{2p}, \dots, G_v c^{(p-1)p}\}$ is an orbit of $\Omega_1(G)$ that is also a part of B_0 . We can label this part by $\{1, g^p, g^{2p}, \dots, g^{(p-1)p}\}$. Note that N acts on $V(G) = N$ by right multiplication. So $\langle g^p \rangle$ fixes each of the p^2 parts of B_0 . It then follows that $K/\langle g^p \rangle \cong C_{p^2}$. So we may assume that $K = \langle g^p \rangle \times \langle g_1 \rangle \cong C_p \times C_{p^2}$. Then $N = \langle g \rangle \langle g_1 \rangle$, and N is metacyclic as $p > 2$. (Note that, by [10, III, 11.5], if $G = \langle a \rangle \langle b \rangle$ is a p -group with $p > 2$, then G is metacyclic.) This implies that Γ is a metacirculant Cayley graph.

Now assume that $c^p \in G_v$. Then c has order p^2 . Assume that $v \in B_0$. We may label the vertices in B_0 in the following way: $v = (d^0, 0)$ and $(d^i, 0) = v^{G_v d^i}$ for $i \in \mathbb{Z}_{p^3}$. Note that $\Gamma[B_0] \cong \text{Cyc}_{p^2}[pK_1]$ and $\Omega_1(G)$ is the kernel of M acting on the p^2 parts of $\text{Cyc}_{p^2}[pK_1]$. As $M = G_v \langle d \rangle$, we have $(d^0, 0)^{\Omega_1(G)} = \{(d^{kp^2}, 0) \mid k \in \mathbb{Z}_p\}$. Since v is adjacent to v^d , $(d^0, 0)$ is adjacent to $(d^1, 0)$. Hence $(d^0, 0)$ is adjacent to all vertices in $(d^1, 0)^{\Omega_1(G)} = \{(d^{1+kp^2}, 0) \mid k \in \mathbb{Z}_p\}$. Similarly, since v is adjacent to $v^{d^{-1}}$, $(d^0, 0)$ is adjacent to all vertices in $(d^{-1}, 0)^{\Omega_1(G)} = \{(d^{-1-kp^2}, 0) \mid k \in \mathbb{Z}_p\}$. Now the edge set of $\Gamma[B_0]$ is $E_0 = \{(d^i, 0), (d^{i+1+kp^2}, 0)\}, \{(d^i, 0), (d^{i-1-kp^2}, 0)\} \mid i \in \mathbb{Z}_{p^3}, k \in \mathbb{Z}_p\}$.

Since c cyclically permutes the p orbits of M , we may assume that $B_j = B_0^{c^j}$ for $j \in \mathbb{Z}_p$. Since v is adjacent to v^c , $(v, v^c, v^{c^2}, \dots, v^{c^{p-1}})$ is a cycle of length p . Without loss of generality we may assume $(d^0, j) = v^{c^j}$ for $j \in \mathbb{Z}_p$ and $(d^i, j) = (d^0, j)^{d^i}$ for $i \in \mathbb{Z}_{p^3}$. We now have $B_j = \{(d^i, j) \mid i \in \mathbb{Z}_{p^3}\}$ for $j \in \mathbb{Z}_p$ and $(d^i, j) \sim (d^i, j+1)$ for $i \in \mathbb{Z}_{p^3}, j \in \mathbb{Z}_p$.

Since $(d^0, 0)^{c^j} = (d^0, j)$, the set of neighbours of (d^0, j) in B_j consists of the two orbits of

$\Omega_1(G)$ containing $(d, 0)^{c^j}$ and $(d^{-1}, 0)^{c^j}$ respectively, namely,

$$((d^0, 0)^{dc^j})^{\Omega_1(G)} \cup ((d^0, 0)^{d^{-1}c^j})^{\Omega_1(G)} = ((d^0, 0)^{c^j})^{c^{-j}dc^j\Omega_1(G)} \cup ((d^0, 0)^{c^j})^{c^{-j}d^{-1}c^j\Omega_1(G)}.$$

Since $M/\Omega_1(G)$ is a normal cyclic subgroup of $G/\Omega_1(G)$ of order p^2 , we have $c^{-1}dc\Omega_1(G) = d^{kp+1}\Omega_1(G)$ for some $k \in \mathbb{Z}_p$. Since $G' \not\leq \Omega_1(G)$ (as $G' \cong C_{p^2}$), we have $k \in \mathbb{Z}_p^*$. Set $\lambda = kp + 1$. Then $c^{-j}dc^j\Omega_1(G) = d^{\lambda^j}\Omega_1(G)$. It follows that the right-hand side of the equation above is $\{(d^{\lambda^j+1+kp^2}, j), (d^{-\lambda^j-1-kp^2}, j) \mid i \in \mathbb{Z}_{p^3}, k \in \mathbb{Z}_p\}$. So the edge set of $\Gamma[B_j]$ is

$$E_j = \{(d^i, j), (d^{i+\lambda^i+kp^2}, j)\}, \{(d^i, j), (d^{i-\lambda^i-kp^2}, j)\} \mid i \in \mathbb{Z}_{p^3}, k \in \mathbb{Z}_p\}.$$

Now we can see that $\Gamma \cong \text{MP}_{p^3, p^2, p^2, \lambda}$. Since $\lambda = kp + 1$, λ must be an element of $\mathbb{Z}_{p^3}^*$ of order p^2 . This completes the proof of Theorem 1.3. \square

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